WESLEY COLLEGE
By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST
SEMESTER TWO 2017
TEST 3: Applications of Calculus

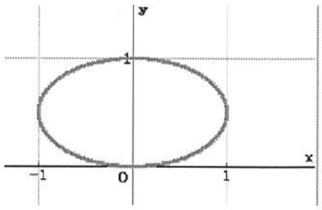
Name: ______________________

Monday 15th August | Time: 50 minutes | Mark /45

Section 1 – Calculator free ~~26~~ 25 marks

1. [~~4~~ 5 marks – ~~3~~ 4 and 1]

The curve defined by the equations $\begin{cases} x(t) = \sin 2t \\ y(t) = \cos^2 t \end{cases}$ for $0 \le t \le 2\pi$



generates the ellipse shown.

(a) Show that $\dfrac{dy}{dx} = -\dfrac{1}{2}\tan 2t$

$$\frac{dx}{dt} = 2\cos 2t$$

$$\frac{dy}{dt} = 2\cos t \cdot (-\sin t) = -\sin 2t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{-\sin 2t}{2\cos 2t} = -\frac{1}{2}\tan 2t$$

(b) What is the slope of this curve at the point where $t = \dfrac{\pi}{8}$?

$$\frac{dy}{dx} = -\frac{1}{2}\tan\frac{\pi}{4} = -\frac{1}{2}$$

2. [4 marks]

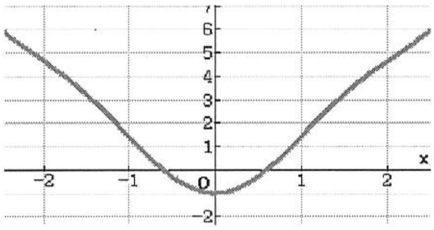
If $y = \cos^3 x$ and $\dfrac{dy}{dt} = 2$, determine the rate of change of x when $x = \dfrac{\pi}{6}$

$$\frac{dy}{dt} = 3\cos^2 x\,(-\sin x)\,\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{2}{3 \cdot \frac{3}{4} \cdot -\frac{1}{2}} = -\frac{16}{9}$$

3. [4 marks – 2 and 2]

$f(x) = x^2 - \cos 2x$ is an even function, symmetric about the y-axis, as shown.



(a) Show clearly that $\int_0^{\frac{\pi}{2}} f(x)\,dx = \dfrac{\pi^3}{24}$

$$\int_0^{\frac{\pi}{2}} x^2 - \cos 2x \; dx = \frac{x^3}{3} - \frac{\sin 2x}{2} \Bigg|_0^{\frac{\pi}{2}} = \frac{\pi^3}{24} - 0 - 0 = \frac{\pi^3}{24}$$

(b) Evaluate $A < 0$ and B so that $\int_A^{\frac{\pi}{2}} B\,f(x)\,dx = \pi^3$

By symmetry $A = -\dfrac{\pi}{2}$

& $B = 12$

4. [~~4~~ 5 marks – 3 and ~~1~~ 2]

(a) Calculate $\pi \int_{\cancel{-\frac{\pi}{4}}\ 0}^{\frac{\pi}{4}} \cos^2\theta - \sin^2\theta \; d\theta$

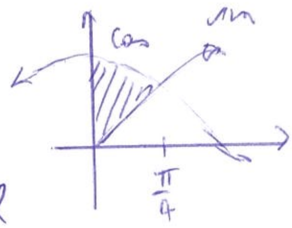
$$= \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta \; d\theta$$

$$= \pi \cdot \frac{\sin 2\theta}{2} \Bigg|_{\cancel{-\frac{\pi}{4}}\ 0}^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2} - \left(\cancel{\sin\frac{\pi}{4}}\ 0\right) = \frac{\pi}{2}$$

(b) Describe, terms of the curves $y = \cos\theta$ and $y = \sin\theta$, the quantity represented by your calculation in part (a).

region between $y = \sin\theta$ + $y = \cos\theta$ between $\theta = 0$ & $\theta = \frac{\pi}{4}$ rotated through $360°$ about the x axis generates this integral as the volume.



5. [~~10~~ 7 marks – ~~4~~ 3, ~~2~~ 1 and ~~4~~ 3]

Determine each of the integrals given:

(a) $\int \frac{x+4}{x^2-2x}\,dx$ where $\frac{x+4}{x^2-2x} = \frac{A}{x} + \frac{B}{x-2}$

$A(x-2) + Bx = x+4$

$\Rightarrow A = -2, \quad B = 3$

$\therefore \int \frac{x+4}{x^2-2x}\,dx = \int \frac{-2}{x} + \frac{3}{x-2}\,dx$

$= -2\ln|x| + 3\ln|x-2| + c$

or $= \ln\left|\frac{(x-2)^3}{x^2}\right| + c$

(b) $\int \frac{\cos(\ln x)}{x}\,dx$

$= \sin(\ln x) + c$

(c) Use the substitution $t = 2 + \cos x$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{2+\cos x}\,dx$ in simplest exact form.

$dt = -\sin x\,dx$

$\Rightarrow \int = \int_3^2 \frac{-dt}{t}$

$= \ln|t| \Big|_2^3$

$= \ln\frac{3}{2}$

Section 2 – Calculator assumed ~~19~~ 20 marks Name:________________________

6. 9 [~~8~~ marks – 3, 2 and ~~3~~] 4

The equation of a curve in the plane is given by $x^2 + 3y^2 + 2xy = 12$

(a) Derive an expression, in terms of x and y, for $\frac{dy}{dx}$

$$2x + 6y\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx}(2x+6y) = -2x - 2y$$

$$\therefore \frac{dy}{dx} = -\frac{x+y}{x+3y}$$

(b) Find the equation of the tangent at the point $(0,2)$

$$\left.\frac{dy}{dx}\right|_{(0,2)} = -\frac{1}{3}$$

$\Rightarrow$ tangent is $y = -\frac{x}{3} + 2$

(c) At which points on the curve is the tangent parallel to the y axis?

$\frac{dy}{dx}$ undefined

$$\Rightarrow x + 3y = 0$$

$$x = -3y$$

$$9y^2 + 3y^2 - 6y^2 = 12$$

$$\Rightarrow y^2 = 2$$

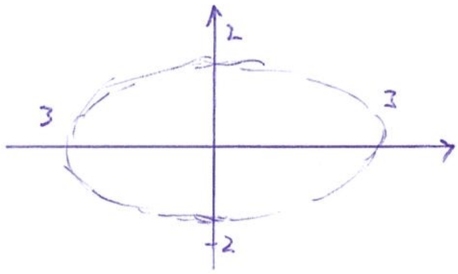
$$y = \pm\sqrt{2}$$

points are $(-3\sqrt{2}, \sqrt{2})$ & $(3\sqrt{2}, -\sqrt{2})$

7. [~~5~~ 6 marks – ~~2~~ 3, 2 and 1]

Compare the volumes generated when the ellipse with equation $\frac{x^2}{9}+\frac{y^2}{4}=1$ is revolved ~~through 360°~~ about the x axis and then the y axis.

Briefly explain why the larger value is larger.



$$V_x = \pi \int_{-3}^{3} 4 - \frac{4x^2}{9} \, dx$$

$$= 2\pi \left(4x - \frac{4x^3}{27} \right) \Big|_0^3$$

$$= 2\pi \left(12 - 4 \right)$$

$$= 16\pi \text{ units}^3$$

$$V_y = \pi \int_{-2}^{2} 9 - \frac{9y^2}{4} \, dy$$

$$= 2\pi \left(9y - \frac{3y^3}{4} \right) \Big|_0^2$$

$$= 2\pi \left(18 - 6 \right)$$

$$= 24\pi \text{ units}^3$$

V_y is greater because radius of outer parts is greater & since Vol ∝ r^3, V_y is larger.

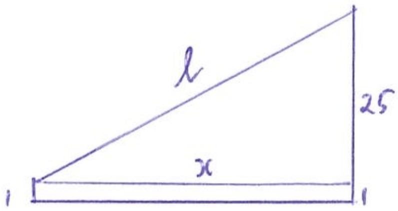
5

8. [~~6~~ marks]

David is flying a kite, which maintains a constant height of 26 m.

The string from David's hand, 1 m above the ground, to the kite is taut (i.e. forms a straight line) and he is releasing this string at a rate of 1.2 m per second.

Describe the motion of the kite when the length of the string is 65 m.



25, 60, 65

5, 12, 13

Find $\frac{dx}{dt}$ at $l = 65$, $x = 60$.

$$l^2 = x^2 + 25^2$$

$$2l \cdot \frac{dl}{dt} = 2x \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{l}{x} \cdot \frac{dl}{dt}$$

$$= \frac{65}{60} \cdot \times 1.2$$

$$= 1.3 \text{ m sec}^{-1}$$

i.e. moving horizontally away from David at 1.3 m sec^{-1}